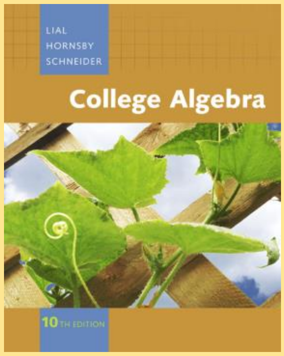
3 Polynomial and Rational Functions

Sections 3.5–3.6



3 Polynomial and Rational Functions

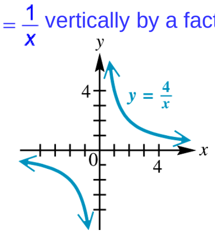
3.5 Rational Functions: Graphs, Applications, and Models

The Reciprocal Function $f(x)=\frac{1}{x}$ ▪ The Function $f(x)=\frac{1}{x^2}$ ▪ Asymptotes ▪ Steps for Graphing Rational Functions ▪ Rational Function Models

3.5 Example 1 Graphing a Rational Function (page 360)

Graph $y=\frac{4}{x}$. Give the domain and range.

The expression $\frac{4}{x}$ can be written as $4\left(\frac{1}{x}\right)$, indicating that the graph can be obtained by stretching the graph of $y=\frac{1}{x}$ vertically by a factor of 4.



Domain: $(-\infty,0)\cup(0,\infty)$

Range: $(-\infty,0)\cup(0,\infty)$

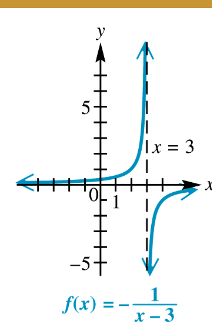
3.5 Example 2 Graphing a Rational Function (page 361)

Graph $f(x)=-\frac{1}{x-3}$. Give the domain and range.

Shift the graph of $y=\frac{1}{x}$ three units to the right, then reflect the graph across the x-axis to obtain the graph of $f(x)=-\frac{1}{x-3}$.

The horizontal asymptote is $y = 0$, the x-axis. The vertical asymptote is $x = 3$.

3.5 Example 2 Graphing a Rational Function (cont.)



Domain: $(-\infty,3)\cup(3,\infty)$

Range: $(-\infty,0)\cup(0,\infty)$

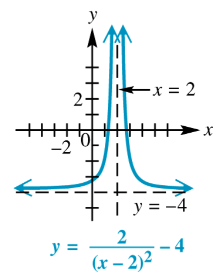
3.5 **Example 3** Graphing a Rational Function (page 363)

Graph $f(x) = \dfrac{2}{(x-2)^2} - 4$. Give the domain and range.

To obtain the graph of $f(x) = \frac{2}{(x-2)^2} - 4$, shift the graph of $f(x) = \frac{1}{x^2}$ two units to the right, stretch the graph vertically by a factor of two, and then shift the graph four units down.

The vertical asymptote is $x = 2$. The horizontal asymptote is $y = -4$.

3.5 **Example 3** Graphing a Rational Function (cont.)



Domain:
$(-\infty, 2) \cup (2, \infty)$
Range:
$(-4, \infty)$

3.5 **Example 4(a)** Finding Asymptotes of Rational Functions (page 364)

Find all asymptotes of the function $f(x) = \dfrac{x+3}{x^2-16}$.

To find the vertical asymptotes, set the denominator equal to 0 and solve.

$$x^2 - 16 = 0 \Rightarrow (x-4)(x+4) = 0 \Rightarrow x = \pm 4$$

The equations of the vertical asymptotes are $x = -4$ and $x = 4$.

The numerator has lower degree than the denominator, so the horizontal asymptote is $y = 0$.

3.5 **Example 4(b)** Finding Asymptotes of Rational Functions (page 364)

Find all asymptotes of the function $f(x) = \dfrac{3x-4}{2x+1}$.

To find the vertical asymptotes, set the denominator equal to 0 and solve.

$$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$$

The equation of the vertical asymptote is $x = -\frac{1}{2}$.

The numerator and the denominator have the same degree, so the equation of the horizontal asymptote is $y = \frac{3}{2}$.

3.5 **Example 4(c)** Finding Asymptotes of Rational Functions (page 364)

Find all asymptotes of the function $f(x) = \dfrac{2x^2+5}{x-3}$.

To find the vertical asymptotes, set the denominator equal to 0 and solve.

$$x - 3 = 0 \Rightarrow x = 3$$

The equation of the vertical asymptote is $x = 3$.

Since the degree of the numerator is exactly one more than the denominator, there is an oblique asymptote.

3.5 **Example 4(c)** Finding Asymptotes of Rational Functions (cont.)

Divide the numerator by the denominator and, disregarding the remainder, set the rest of the quotient equal to y to obtain the equation of the asymptote.

$$\begin{array}{r|l} & 2x + 6 \\ \hline x - 3 & 2x^2 + 0x + 5 \\ & \underline{2x^2 - 6x} \\ & \quad 6x + 5 \\ & \quad \underline{6x - 18} \\ & \quad\quad 23 \end{array}$$

The equation of the oblique asymptote is $y = 2x + 6$.

3.5 **Example 5** Graphing a Rational Function with the *x*-axis as Horizontal Asymptote (page 366)

Graph $f(x)=\dfrac{x-2}{x^2-x-6}$.

Step 1: Find the vertical asymptotes by setting the denominator equal to 0 and solving.

$$x^2-x-6=0 \Rightarrow (x-3)(x+2)=0 \Rightarrow x=3 \text{ or } x=-2$$

Step 2: Find the horizontal asymptote.

The numerator has lower degree than the denominator, so the horizontal asymptote is $y = 0$.

3.5 **Example 5** Graphing a Rational Function with the *x*-axis as Horizontal Asymptote (cont.)

Step 3: Find the *y*-intercept.

$$f(0)=\frac{0-2}{0^2-0-6}=\frac{1}{3}$$

Step 4: Find the *x*-intercept.

$$\frac{x-2}{x^2-x-6}=0 \Rightarrow x-2=0 \Rightarrow x=2$$

3.5 **Example 5** Graphing a Rational Function with the *x*-axis as Horizontal Asymptote (cont.)

Step 5: Determine whether the graph intersects its horizontal asymptote.

Solve $f(x) = 0$ (*y*-value of the horizontal asymptote).

Since the horizontal asymptote is the *x*-axis, the solution of this equation was found in Step 4. The graph intersects its horizontal asymptote at (2, 0).

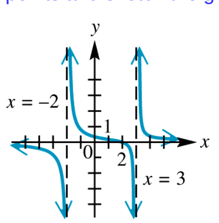
3.5 **Example 5** Graphing a Rational Function with the *x*-axis as Horizontal Asymptote (cont.)

Step 6: Choose a test point in each of the intervals determined by the *x*-intercept and the vertical asymptotes to determine how the graph behaves.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below *x*-Axis
$(-\infty,-2)$	-3	$-\frac{5}{6}$	Negative	Below
$(-2, 2)$	0	$\frac{1}{3}$	Positive	Above
$(2, 3)$	$\frac{5}{2}$	$-\frac{2}{9}$	Negative	Below
$(3,\infty)$	4	$\frac{1}{3}$	Positive	Above

3.5 **Example 5** Graphing a Rational Function with the *x*-axis as Horizontal Asymptote (cont.)

Step 7: Plot the points and sketch the graph.



$$f(x)=\frac{x-2}{x^2-x-6}$$

3.5 **Example 6** Graphing a Rational Function That Does Not Intersect its Horizontal Asymptote (page 367)

Graph $f(x)=\dfrac{4x-2}{x+1}$.

Step 1: Find the vertical asymptote by setting the denominator equal to 0 and solving.

$$x+1=0 \Rightarrow x=-1$$

Step 2: Find the horizontal asymptote.

The numerator has the same degree as the denominator, so the horizontal asymptote is $y = 4$.

3.5 **Example 6** Graphing a Rational Function That Does Not Intersect its Horizontal Asymptote (cont.)

Step 3: Find the *y*-intercept.

$$f(0)=\frac{4(0)-2}{0+1}=-2$$

Step 4: Find the *x*-intercept.

$$\frac{4x-2}{x+1}=0\Rightarrow 4x-2=0\Rightarrow x=\tfrac{1}{2}$$

3.5 **Example 6** Graphing a Rational Function That Does Not Intersect its Horizontal Asymptote (cont.)

Step 5: Determine whether the graph intersects its horizontal asymptote.

Solve *f*(*x*) = 4 (*y*-value of the horizontal asymptote).

$$f(x)=\frac{4x-2}{x+1}=4\Rightarrow 4x-2=4x+4\Rightarrow -2=4 \text{ False}$$

The graph does not intersect its horizontal asympote.

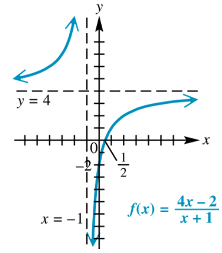
3.5 **Example 6** Graphing a Rational Function That Does Not Intersect its Horizontal Asymptote (cont.)

Step 6: Choose a test point in each of the intervals determined by the *x*-intercept and the vertical asymptote to determine how the graph behaves.

Interval	Test Point	Value of *f* (*x*)	Sign of *f* (*x*)	Graph Above or Below *x*-Axis
$(-\infty,-1)$	–3	7	Positive	Above
$(-1,\frac{1}{2})$	0	–2	Negative	Below
$(\frac{1}{2},\infty)$	1	1	Positive	Above

3.5 **Example 6** Graphing a Rational Function That Does Not Intersect its Horizontal Asymptote (cont.)

Step 7: Plot the points and sketch the graph.



3.5 **Example 7** Graphing a Rational Function That Intersects its Horizontal Asymptote (page 368)

Graph $f(x)=\dfrac{2x^2+3x-4}{x^2+6x+9}$.

Step 1: Find the vertical asymptote by setting the denominator equal to 0 and solving.

$$x^2+6x+9=0\Rightarrow (x+3)^2=0\Rightarrow x=-3$$

Step 2: Find the horizontal asymptote.

The numerator has the same degree than the denominator, so the horizontal asymptote is *y* = 2.

3.5 **Example 7** Graphing a Rational Function That Intersects its Horizontal Asymptote (cont.)

Step 3: Find the *y*-intercept.

$$f(0)=\frac{2(0)^2+3(0)-4}{0^2+6(0)+9}=-\frac{4}{9}$$

Step 4: Find the *x*-intercepts.

$$\frac{2x^2+3x-4}{x^2+6x+9}=0\Rightarrow 2x^2+3x-4=0$$

$$x=\frac{-3\pm\sqrt{3^2-4(2)(-4)}}{2(2)}$$

$$=\frac{-3\pm\sqrt{41}}{4}\Rightarrow x\approx -2.35 \text{ or } x\approx .85$$

3.5 **Example 7** Graphing a Rational Function That Intersects its Horizontal Asymptote (cont.)

Step 5: Determine whether the graph intersects its horizontal asymptote.

Solve $f(x) = 2$ (y-value of the horizontal asymptote).

$$f(x) = \frac{2x^2+3x-4}{x^2+6x+9} = 2$$

$$2x^2+3x-4 = 2x^2+12x+18$$

$$-22 = 9x \Rightarrow -\frac{22}{9} = x$$

The graph intersects its horizontal asymptote at $\left(-\frac{22}{9}, 2\right)$.

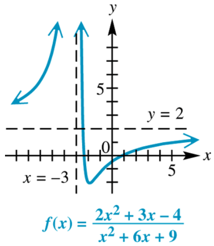
3.5 **Example 7** Graphing a Rational Function That Intersects its Horizontal Asymptote (cont.)

Step 6: Choose a test point in each of the intervals determined by the x-intercepts and the vertical asymptote to determine how the graph behaves.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	–4	16	Positive	Above
$(-3, -2.35)$	–2.5	4	Positive	Above
$(-2.35, .85)$	0	$-\frac{4}{9}$	Negative	Below
$(.85, \infty)$	2	$\frac{2}{5}$	Positive	Above

3.5 **Example 7** Graphing a Rational Function That Intersects its Horizontal Asymptote (cont.)

Step 7: Plot the points and sketch the graph.



3.5 **Example 8** Graphing a Rational Function With an Oblique Asymptote (page 369)

Graph $f(x) = \frac{x^2+x}{x-1}$.

Step 1: Find the vertical asymptote by setting the denominator equal to 0 and solving.

$$x - 1 = 0 \Rightarrow x = 1$$

Step 2: Since the degree of the numerator is exactly one more than the denominator, there is an oblique asymptote.

3.5 **Example 8** Graphing a Rational Function With an Oblique Asymptote (cont.)

Divide the numerator by the denominator and, disregarding the remainder, set the rest of the quotient equal to y to obtain the equation of the asymptote.

$$\begin{array}{r} x + 2 \\ x-1\overline{)\,x^2 + x} \\ \underline{x^2 - x} \\ 2x \\ \underline{2x - 2} \\ 2 \end{array}$$

The equation of the oblique asymptote is $y = x + 2$.

3.5 **Example 8** Graphing a Rational Function With an Oblique Asymptote (cont.)

Step 3: Find the y-intercept.

$$f(0) = \frac{0^2+0}{0-1} = 0$$

Step 4: Find the x-intercepts.

$$\frac{x^2+x}{x-1} = 0 \Rightarrow x^2 + x = 0 \Rightarrow x(x+1) = 0 \Rightarrow x = 0 \text{ or } x = -1$$

3.5 **Example 8** Graphing a Rational Function With an Oblique Asymptote (cont.)

Step 5: Determine whether the graph intersects its oblique asymptote.

Solve $f(x) = x + 2$.

$$\frac{x^2+x}{x-1} = x+2$$
$$x^2 + x = (x+2)(x-1)$$
$$x^2 + x = x^2 + x - 2$$
$$0 = 2 \quad \text{False}$$

The graph does not intersect its oblique asympote.

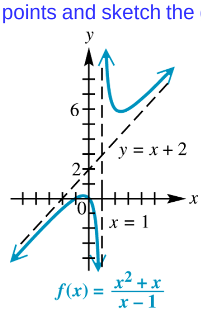
3.5 **Example 8** Graphing a Rational Function With an Oblique Asymptote (cont.)

Step 6: Choose a test point in each of the intervals determined by the *x*-intercepts and the vertical asymptote to determine how the graph behaves.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below *x*-Axis
$(-\infty,-1)$	-4	$-\frac{12}{5}$	Negative	Below
$(-1, 0)$	$-\frac{1}{2}$	$\frac{1}{6}$	Positive	Above
$(0, 1)$	$\frac{1}{2}$	$-\frac{3}{2}$	Negative	Below
$(1,\infty)$	2	6	Positive	Above

3.5 **Example 8** Graphing a Rational Function With an Oblique Asymptote (cont.)

Step 7: Plot the points and sketch the graph.



3.5 **Example 9** Graphing a Rational Function Defined by an Expression That is not in Lowest Terms (page 370)

Graph $f(x) = \frac{x^2 - x - 6}{x-3}$.

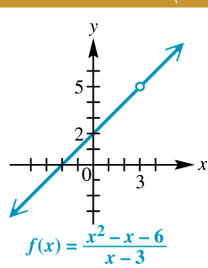
The domain of *f* cannot include 3.

In lowest terms, the function becomes

$$f(x) = \frac{x^2-x-6}{x-3} = \frac{(x-3)(x+2)}{x-3} = x+2,\ x \neq 3$$

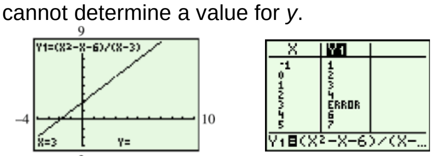
3.5 **Example 9** Graphing a Rational Function Defined by an Expression That is not in Lowest Terms (cont.)

The graph of *f* is the same as the graph of $y = x + 2$, with the exception of the point with *x*-value 3. A "hole" appears in the graph at (3, 5).



3.5 **Example 9** Graphing a Rational Function Defined by an Expression That is not in Lowest Terms (cont.)

If the window of a graphing calculator is set so that an *x*-value of 3 is displayed, then the calculator cannot determine a value for *y*.



Notice the visible discontinuity at $x = 3$. The error message in the table supports the existence of the discontinuity.

3.5 **Example 10** Modeling Traffic Intensity with a Rational Function (page 371)

Vehicles arrive randomly at a parking ramp at an average rate of 3.6 vehicles per minute. The parking attendant can admit 4.0 vehicles per minute. Since arrivals are random, lines form at various times. (*Source*: Mannering, F. and W. Kilareski, *Principles of Highway Engineering and Traffic Analysis*, 2nd ed., John Wiley & Sons, 1998.)

(a) The **traffic intensity** *x* is defined as the ratio of the average arrival rate to the average admittance rate. Determine *x* for this parking ramp.

$$x = \frac{3.6}{4.0} = .9$$

3.5 **Example 10** Modeling Traffic Intensity with a Rational Function (page 371)

(b) The average number of vehicles waiting in line to enter the ramp is given by

$$f(x) = \frac{x^2}{2(1-x)}$$

where $0 \le x < 1$ is the traffic intensity. Compute the average number of vehicles waiting in line to enter the ramp.

$$f(.9) = \frac{x^2}{2(1-x)} = \frac{.9^2}{2(1-.9)} = 4.05$$

The average number of vehicles waiting in line to enter the ramp is 4.05.

3.6 Variation

Direct Variation ▪ Inverse Variation ▪ Combined and Joint Variation

3.6 **Example 1** Solving a Direct Variation Problem (page 380)

At a given average speed, the distance traveled by a vehicle varies directly as the time. If a vehicle travels 156 miles in 3 hours, find the distance it will travel in 5 hours at the same average speed.

Step 1: Since the distance varies directly as the time, ***d* = *kt***.

Step 2: Substitute ***d* = 156** and ***t* = 3** to find *k*.

$$156 = k(3) \Rightarrow k = 52$$

3.6 **Example 1** Solving a Direct Variation Problem (cont.)

Step 3: The relationship between distance and time is ***d* = 52*t***.

Step 4: Solve the equation for ***d*** with ***t* = 5**.

$$d = 52(5) = 260$$

The vehicle will travel 260 miles in 5 hours.

3.6 **Example 2** Solving an Inverse Variation Problem (page 380)

In a certain manufacturing process, the cost of producing a single item varies inversely as the square of the number of items produced. If 100 items are produced, each costs $1.50. Find the cost per item if 250 items are produced.

Step 1: Let *x* represent the number of items produced and *y* represent the cost per item.

$$y = \frac{k}{x^2}$$

3.6 **Example 2** Solving an Inverse Variation Problem (cont.)

Step 2: Substitute ***y* = 1.50** and ***x* = 100** to find *k*.

$$1.50 = \frac{k}{100^2} \Rightarrow k = 15{,}000$$

Step 3: The relationship between *x* and *y* is $y = \frac{15{,}000}{x^2}$.

Step 4: Solve the equation for ***y*** with ***x* = 250**.

$$y = \frac{15{,}000}{250^2} = .24$$

The cost per item will be $0.24.

3.6 **Example 3** Solving a Joint Variation Problem (page 382)

The volume of a cylinder varies directly as the height and the square of the radius. A cylinder with radius 5 cm and height 10 cm has volume 785 cm^3. Find the volume of a cylinder with radius 10 cm and height 15 cm.

Step 1: Since the volume varies directly as the height and the square of the radius, $V = khr^2$.

Step 2: Substitute ***V* = 785, *h* = 10** and ***r* = 5** to find *k*.

$$785 = k(10)\left(5^2\right) \Rightarrow k = \frac{157}{50}$$

3.6 **Example 3** Solving a Direct Variation Problem (cont.)

Step 3: The relationship among the variables is

$$V = \frac{157}{50}hr^2$$

Step 4: Solve the equation for ***V*** with ***h* = 15** and ***r* = 10**.

$$V = \frac{157}{50}(15)\left(10^2\right) = 4710$$

The volume of the cylinder is 4710 cm^3.

3.6 **Example 4** Solving a Combined Variation Problem (page 383)

Vanessa Jones has determined that the number of children enrolled in her day care center varies directly with the amount she spends on advertising per year and inversely with her weekly charge for each child. This year she spent $900 on advertising, charged $150 per child per week, and had 51 children enrolled. Next year, if she spend $1000 on advertising and charges $170 per week, how many children should she expect to enroll?

Let *n* = the number of children, *a* = the amount spent on advertising per year, and *c* = the weekly charge. Then

$$n = \frac{ka}{c}$$

3.6 **Example 4** Solving a Combined Variation Problem (cont.)

Substitute the given values for *n*, *a*, and *c* to determine *k*.

$$51 = \frac{900k}{150} \Rightarrow k = 8.5$$

The relationship among the variables is $n = \frac{8.5a}{c}$.

Solve the equation for ***n*** with ***a* = 1000** and ***c* = 170**.

$$n = \frac{8.5(1000)}{170} = 50$$

She should expect to enroll 50 children.